Svar på opgave 2007-75
Maj 2007

Opgaven:

Lad a, b og c være forskellige reelle tal, der ikke er 0, således at $a + b + c = 0$.

Vis, at

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=3 \quad \text{og} \quad \left(\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}\right)\cdot\left(\frac{c}{a-b}+\frac{a}{b-c}+\frac{b}{c-a}\right)=9\ .$$

I begge tilfælde skal samtlige mellemregninger angives.

Løsning:

a.
Vi benytter, at

$$a = -b-c\ ,\ b = -a-c\ ,\ c = -a-b\ ,$$

og får

$$\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}=\frac{(-b-c)^2}{bc}+\frac{(-a-c)^2}{ac}+\frac{(-a-b)^2}{ab}$$

$$=\frac{b^2+c^2+2bc}{bc}+\frac{a^2+c^2+2ac}{ac}+\frac{a^2+b^2+2ab}{ab}=\frac{b}{c}+\frac{c}{b}+2+\frac{a}{c}+\frac{c}{a}+2+\frac{a}{b}+\frac{b}{a}+2$$

$$=6+\frac{a+b}{c}+\frac{b+c}{a}+\frac{c+a}{b}=6+\frac{-c}{c}+\frac{-a}{a}+\frac{-b}{b}=3\ .$$

b.
Vi udregner hver af de to parenteser hver for sig, idet vi benytter, at c = - a - b. Den første parentes:

$$\frac{a-b}{c}+\frac{b-c}{a}+\frac{c-a}{b}=\frac{ab(a-b)+bc(b-c)+ac(c-a)}{abc}$$

$$=\frac{ab(a-b)-b(a+b)(b+a+b)-a(a+b)(-a-b-a)}{-ab(a+b)}$$

$$=\frac{ab(a-b)-b(a+b)(a+2b)+a(a+b)(2a+b)}{-ab(a+b)}$$

$$= \frac{ab(a-b)+(a+b)(-b(a+2b)+a(2a+b))}{-ab(a+b)}$$

$$= \frac{ab(a-b)}{-ab(a+b)} + \frac{(a+b)(-ab-2b^2+2a^2+ab)}{-ab(a+b)}$$

$$= -\frac{a-b}{a+b} - \frac{2a^2-2b^2}{ab} = -\frac{a-b}{a+b} - \frac{2(a+b)(a-b)}{ab}$$

$$= -\frac{(a-b)ab+2(a+b)(a-b)(a+b)}{ab(a+b)} = -\frac{(a-b)(ab+2a^2+2b^2+4ab)}{ab(a+b)}$$

$$= -\frac{(a-b)(2a^2+2b^2+5ab)}{ab(a+b)}$$

Den anden parentes:

$$\frac{c}{a-b} + \frac{a}{b-c} + \frac{b}{c-a} = \frac{-a-b}{a-b} + \frac{a}{b+a+b} + \frac{b}{-a-b-a}$$

$$= -\frac{a+b}{a-b} + \frac{a}{2b+a} - \frac{b}{2a+b} = -\frac{a+b}{a-b} + \frac{a(2a+b)-b(2b+a)}{(2b+a)(2a+b)}$$

$$= -\frac{a+b}{a-b} + \frac{2a^2+ab-2b^2-ab}{(2b+a)(2a+b)} = -\frac{a+b}{a-b} + \frac{2(a^2-b^2)}{(2b+a)(2a+b)}$$

$$= -\frac{a+b}{a-b} + \frac{2(a+b)(a-b)}{(2b+a)(2a+b)} = (a+b)\left(\frac{2(a-b)}{(2b+a)(2a+b)} - \frac{1}{a-b}\right)$$

$$= (a+b)\cdot\frac{2(a-b)(a-b)-(2b+a)(2a+b)}{(2b+a)(2a+b)}$$

$$= (a+b)\cdot\frac{2a^2+2b^2-4ab-(4ab+2b^2+2a^2+ab)}{(2b+a)(2a+b)(a-b)}$$

$$= (a+b)\cdot\frac{-9ab}{(2b+a)(2a+b)(a-b)} = \frac{-9ab(a+b)}{(2b+a)(2a+b)(a-b)}$$

Nu giver multiplikation af de to parenteser:

$$-\frac{(a-b)(2a^2+2b^2+5ab)}{ab(a+b)}\cdot\frac{-9ab(a+b)}{(2b+a)(2a+b)(a-b)}$$

$$= \frac{-(2a^2+2b^2+5ab)\cdot(-9)}{(2b+a)(2a+b)} = \frac{(2a^2+2b^2+5ab)\cdot 9}{2a^2+2b^2+5ab} = 9$$