Svar på opgave 228 (Marts 2006)

Opgave:

Vis følgende trigonometriske identiteter:

a. $(1 - \cot 22°)(1 - \cot 23°) = 2$

b. $\dfrac{1}{\cos 6°} + \dfrac{1}{\sin 24°} + \dfrac{1}{\sin 48°} = \dfrac{1}{\sin 12°}$

c. $\tan 37\tfrac{1}{2}° = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2$.

Besvarelse:

a. Vi skal vise, at

$$(1 - \cot 22°)(1 - \cot 23°) = 2$$

1. metode.
Vi får, at

$$k = (1 - \cot 22°)(1 - \cot 23°) = \left(1 - \frac{\cos 23°}{\sin 23°}\right) \cdot \left(1 - \frac{\cos 22°}{\sin 22°}\right)$$
$$= \frac{\sin 23° - \cos 23°}{\sin 23°} \cdot \frac{\sin 22° - \cos 22°}{\sin 22°} \ .$$

Nu gælder, at

$$\sin x - \cos x = \sqrt{2} \cdot \sin(x - 45°) \ ,$$

så vi får

$$k = \frac{\sqrt{2} \cdot \sin(23° - 45°) \cdot \sqrt{2} \cdot \sin(22° - 45°)}{\sin 23° \cdot \sin 22°}$$
$$= \frac{2 \cdot \sin(-22°) \cdot \sin(-23°)}{\sin 23° \cdot \sin 22°} = 2 \ .$$

I almindelighed gælder, at hvis $x + y = 45°$, så er $(1 - \cot x)(1 - \cot y) = 2$.

2. metode.
Additionsformlerne for cot giver

$$\frac{\cot 22° \cdot \cot 23° - 1}{\cot 22° + \cot 23°} = \cot(22° + 23°) = \cot 45° = 1 \ ,$$

Så

$$\cot 22° \cdot \cot 23° - 1 = \cot 22° + \cot 23°$$
$$\Leftrightarrow \quad -1 - \cot 22° - \cot 23° + \cot 22° \cdot \cot 23° = 0$$
$$\Leftrightarrow \quad 1 - \cot 22° - \cot 23° + \cot 22° \cdot \cot 23° = 2$$
$$\Leftrightarrow \quad (1 - \cot 22°)(1 - \cot 23°) = 2 \ .$$

b. Vi skal vise, at

$$\frac{1}{\cos 6^\circ} + \frac{1}{\sin 24^\circ} + \frac{1}{\sin 48^\circ} = \frac{1}{\sin 12^\circ} \ .$$

1. metode.

Vi omskriver ligningen således:

$$\frac{1}{\cos 6^\circ} + \frac{1}{\sin 24^\circ} + \frac{1}{\sin 48^\circ} = \frac{1}{\sin 12^\circ} \ \Leftrightarrow \ \frac{1}{\sin 96^\circ} + \frac{1}{\sin 48^\circ} = \frac{1}{\sin 12^\circ} - \frac{1}{\sin 24^\circ}$$

$$\Leftrightarrow \ \frac{\sin 48^\circ + \sin 96^\circ}{\sin 96^\circ \cdot \sin 48^\circ} = \frac{\sin 24^\circ - \sin 12^\circ}{\sin 12^\circ \cdot \sin 24^\circ}$$

$$\Leftrightarrow \ \frac{\sin 48^\circ + 2\sin 48^\circ \cos 48^\circ}{\sin 96^\circ \cdot \sin 48^\circ} = \frac{2\sin 12^\circ \cos 12^\circ - \sin 12^\circ}{\sin 12^\circ \cdot \sin 24^\circ}$$

$$\Leftrightarrow \ \frac{1 + 2\cos 48^\circ}{\sin 96^\circ} = \frac{2\cos 12^\circ - 1}{\sin 24^\circ}$$

$$\Leftrightarrow \ \sin 24^\circ + 2\cos 48^\circ \cdot \sin 24^\circ = 2\cos 12^\circ \cdot \sin 96^\circ - \sin 96^\circ$$

$$\Leftrightarrow \ \sin 24^\circ + (\sin(48^\circ+24^\circ) - \sin(48^\circ-24^\circ)) = (\sin(96^\circ+12^\circ) + \sin(96^\circ-12^\circ)) - \sin 96^\circ$$

$$\Leftrightarrow \ \sin 72^\circ = \sin 108^\circ + \sin 84^\circ - \sin 96^\circ \ ,$$

og dette er sandt, fordi

$$\sin 72^\circ = \sin 108^\circ \quad \text{og} \quad \sin 84^\circ = \sin 96^\circ \ .$$

2. metode.

Vi omskriver sådan:

$$\frac{1}{\cos 6^\circ} + \frac{1}{\sin 24^\circ} = \frac{\sin 24^\circ + \cos 6^\circ}{\cos 6^\circ \cdot \sin 24^\circ} = \frac{\sin 24^\circ + \sin 84^\circ}{\cos 6^\circ \cdot \sin 24^\circ}$$

$$= \frac{2 \cdot \sin 54^\circ \cdot \cos 30^\circ}{\frac{1}{2}(\sin 18^\circ + \sin 30^\circ)} = 2\sqrt{3} \cdot \frac{\sin 54^\circ}{\sin 18^\circ + \frac{1}{2}} = 2\sqrt{3} \cdot \frac{\cos 36^\circ}{\sin 18^\circ + \frac{1}{2}} \ .$$

Tilsvarende er

$$\frac{1}{\sin 12^\circ} + \frac{1}{\sin 48^\circ} = \frac{\sin 48^\circ - \sin 12^\circ}{\sin 12^\circ \cdot \sin 48^\circ} = \frac{\sin 48^\circ - \sin 12^\circ}{\frac{1}{2}(\cos 36^\circ - \cos 60^\circ)}$$

$$= \frac{2 \cdot \cos 30^\circ \cdot \sin 18^\circ}{\frac{1}{2}(\cos 36^\circ - \cos 60^\circ)} = 2\sqrt{3} \cdot \frac{\sin 18^\circ}{\cos 36^\circ - \frac{1}{2}} \ .$$

Vi skal altså vise, at

$$\frac{\cos 36^\circ}{\sin 18^\circ + \frac{1}{2}} = \frac{\sin 18^\circ}{\cos 36^\circ - \frac{1}{2}} \quad \text{eller} \quad \cos^2 36^\circ - \tfrac{1}{2}\cos 36^\circ = \sin^2 18^\circ + \tfrac{1}{2}\sin 18^\circ \ .$$

Da $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ og $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ er dette ensbetydende med

$$\tfrac{1}{2}(1 + \cos 72^\circ) - \tfrac{1}{2}\cos 36^\circ = \tfrac{1}{2}(1 - \cos 36^\circ) + \tfrac{1}{2}\sin 18^\circ$$

$$\Leftrightarrow \ \cos 72^\circ - \cos 36^\circ = \sin 18^\circ - \cos 36^\circ \ \Leftrightarrow \ \cos 72^\circ = \sin 18^\circ \ ,$$

hvilket er sandt.

c. Vi skal vise, at

$$\tan 37\tfrac{1}{2}^\circ = \sqrt{6}+\sqrt{3}-\sqrt{2}-2 \ .$$

Vi skriver sådan:

$$\tan 37\tfrac{1}{2}^\circ = \frac{\sin\frac{75^\circ}{2}}{\cos\frac{75^\circ}{2}} = \frac{\cos\left(90^\circ-\frac{75^\circ}{2}\right)}{\cos\frac{75^\circ}{2}} = \frac{2\cdot\cos\frac{105^\circ}{2}\cdot\sin\frac{15^\circ}{2}}{2\cdot\cos\frac{75^\circ}{2}\cdot\sin\frac{15^\circ}{2}}$$

$$= \frac{\sin 60^\circ - \sin 45^\circ}{\sin 45^\circ - \sin 30^\circ} = \frac{\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}-\frac{1}{2}} = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}-1}$$

$$= (\sqrt{3}-\sqrt{2})(\sqrt{2}+1) = \sqrt{6}+\sqrt{3}-\sqrt{2}-2 \ .$$

Der er modtaget 6 besvarelser.