Svar på opgave 305 (December 2013)

Opgave:

Vis at

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots+\frac{1}{n^3} \;<\; \frac{5}{4}\,.$$

Besvarelse:

Vi sætter

$$S=1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\cdots+\frac{1}{n^3}\,.$$

Vi har, at

$$n^3 \;>\; n^3 - n = n(n^2 - 1) = (n - 1)\cdot n\cdot(n + 1)\;\;,$$

og desuden at

$$\frac{1}{(n-1)\cdot n\cdot(n+1)}=\frac{1}{2}\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\right)\;,$$

hvoraf vurderingen

$$\frac{1}{n^3} \;<\; \frac{1}{2}\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\right)\;.$$

Dermed er

$$S \;<\; 1+\frac{1}{2}\left(1-1+\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right)+\cdots+\frac{1}{2}\left(\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\right)$$

$$=1+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{n}+\frac{1}{n+1}\right)=\frac{5}{4}+\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n}\right) \;<\; \frac{5}{4}\,.$$